

# DILATION THEORY YESTERDAY AND TODAY

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ABSTRACT. Paul Halmos' work in dilation theory began with a question and its answer: Which operators on a Hilbert space  $H$  can be extended to normal operators on a larger Hilbert space  $K \supseteq H$ ? The answer is interesting and subtle.

The idea of representing operator-theoretic structures in terms of conceptually simpler structures acting on larger Hilbert spaces has become a central one in the development of operator theory and, more generally, noncommutative analysis. The work continues today. In this article we summarize some of these diverse results and their history.

## 1. PREFACE

What follows is a brief account of the development of dilation theory that highlights Halmos' contribution to the circle of ideas. The treatment is not comprehensive. I have chosen topics that have interested me over the years, while perhaps neglecting others. In order of appearance, the cast includes dilation theory for subnormal operators, operator valued measures and contractions, operator spaces, the role of extensions in dilation theory, commuting sets of operators, and semigroups of completely positive maps. We emphasize Stinespring's theorem, but barely mention the model theory of Sz.-Nagy and Foias or its application to systems theory.

After reflection on the common underpinnings of these results, it seemed a good idea to highlight the role of Banach  $*$ -algebras in their proofs, and I have done that. An appendix is included that summarizes what is needed. Finally, we have tried to make the subject accessible to students by keeping the prerequisites to a minimum; but of course we do assume familiarity with the basic theory of operators on Hilbert spaces and  $C^*$ -algebras.

## 2. ORIGINS

Hilbert spaces are important because positive definite functions give rise to inner products on vector spaces – whose completions are Hilbert spaces – and positive definite functions are found in every corner of mathematics and mathematical physics. This association of a Hilbert space with a positive definite function involves a construction, and like all constructions that begin with objects in one category and generate objects in another category, it is best understood when viewed as a *functor*. We begin by discussing the

properties of this functor in some detail since, while here they are simple and elementary, similar properties will re-emerge later in other contexts.

Let  $X$  be a set and let

$$u : X \times X \rightarrow \mathbb{C}$$

be a complex-valued function of two variables that is *positive definite* in the sense that for every  $n = 1, 2, \dots$ , every  $x_1, \dots, x_n \in X$  and every set  $\lambda_1, \dots, \lambda_n$  of complex numbers, one has

$$(2.1) \quad \sum_{k,j=1}^n u(x_k, x_j) \bar{\lambda}_j \lambda_k \geq 0.$$

Notice that if  $f : X \rightarrow H$  is a function from  $X$  to a Hilbert space  $H$  with inner product  $\langle \cdot, \cdot \rangle$ , then the function  $u : X \times X \rightarrow \mathbb{C}$  defined by

$$(2.2) \quad u(x, y) = \langle f(x), f(y) \rangle, \quad x, y \in X$$

is positive definite. By passing to a subspace of  $H$  if necessary, one can obviously arrange that  $H$  is the closed linear span of the set of vectors  $f(X)$  in the range of  $f$ , and in that case the function  $f : X \rightarrow H$  is said to be *minimal* (for the positive definite function  $u$ ). Let us agree to say that two Hilbert space valued functions  $f_1 : X \rightarrow H_1$  and  $f_2 : X \rightarrow H_2$  are *isomorphic* if there is a unitary operator  $U : H_1 \rightarrow H_2$  such that

$$U(f_1(x)) = f_2(x), \quad x \in X.$$

A simple argument shows that all minimal functions for  $u$  are isomorphic.

For any positive definite function  $u : X \times X \rightarrow \mathbb{C}$ , a self-map  $\phi : X \rightarrow X$  may or may not preserve the values of  $u$  in the sense that

$$u(\phi(x), \phi(y)) = u(x, y), \quad x, y \in X;$$

but when this formula does hold, one would expect that  $\phi$  should acquire a Hilbert space identity. In order to discuss that, let us think of Hilbert spaces as the objects of a category whose morphisms are isometries; thus, a homomorphism from  $H_1$  to  $H_2$  is a linear isometry  $U \in \mathcal{B}(H_1, H_2)$ . Positive definite functions are also the objects of a category, in which a homomorphism from  $u_1 : X_1 \times X_1 \rightarrow \mathbb{C}$  to  $u_2 : X_2 \times X_2 \rightarrow \mathbb{C}$  is a function  $\phi : X_1 \rightarrow X_2$  that preserves the positive structure in the sense that

$$(2.3) \quad u_2(\phi(x), \phi(y)) = u_1(x, y), \quad x, y \in X_1.$$

Given a positive definite function  $u : X \times X \rightarrow \mathbb{C}$ , one can construct a Hilbert space  $H(u)$  and a function  $f : X \rightarrow H(u)$  as follows. Consider the vector space  $\mathbb{C}X$  of all complex valued functions  $\xi : X \rightarrow \mathbb{C}$  with the property that  $\xi(x) = 0$  for all but a finite number of  $x \in X$ . We can define a sesquilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{C}X$  by way of

$$\langle \xi, \eta \rangle = \sum_{x,y \in X} u(x, y) \xi(x) \bar{\eta}(y), \quad \xi, \eta \in \mathbb{C}X,$$

and one finds that  $\langle \cdot, \cdot \rangle$  is positive semidefinite because of the hypothesis on  $u$ . An application of the Schwarz inequality shows that the set

$$N = \{\xi \in \mathbb{C}X : \langle \xi, \xi \rangle = 0\}$$

is in fact a linear subspace of  $\mathbb{C}X$ , so this sesquilinear form can be promoted naturally to an inner product on the quotient  $\mathbb{C}X/N$ . The completion of the inner product space  $\mathbb{C}X/N$  is a Hilbert space  $H(u)$ , and we can define the sought-after function  $f : X \rightarrow H(u)$  as follows:

$$(2.4) \quad f(x) = \delta_x + N, \quad x \in X,$$

where  $\delta_x$  is the characteristic function of the singleton  $\{x\}$ . By construction,  $u(x, y) = \langle f(x), f(y) \rangle$ . Note too that this function  $f$  is *minimal* for  $u$ . While there are many (mutually isomorphic) minimal functions for  $u$ , we fix attention on the minimal function (2.4) that we have constructed.

Given two positive definite functions  $u_k : X_k \times X_k \rightarrow \mathbb{C}$ ,  $k = 1, 2$ , choose a homomorphism from  $u_1$  to  $u_2$ , namely a function  $\phi : X_1 \rightarrow X_2$  that satisfies (2.3). Notice that while the two functions  $f_k : X_k \rightarrow H(u_k)$

$$f_1(x) = \delta_x + N_1, \quad f_2(y) = \delta_y + N_2, \quad x \in X_1, \quad y \in X_2$$

need not be injective, we do have the relations

$$\langle f_2(\phi(x)), f_2(\phi(y)) \rangle_{H(u_2)} = u_2(\phi(x), \phi(y)) = u_1(x, y) = \langle f_1(x), f_1(y) \rangle_{H(u_1)},$$

holding for all  $x, y \in X_1$ . Since  $H(u_1)$  is spanned by  $f_1(X_1)$ , a familiar and elementary argument (that we omit) shows that there is a unique linear isometry  $U_\phi : H(u_1) \rightarrow H(u_2)$  such that

$$(2.5) \quad U_\phi(f_1(x)) = f_2(\phi(x)), \quad x \in X_1.$$

At this point, it is straightforward to verify that the expected composition formulas  $U_{\phi_1} U_{\phi_2} = U_{\phi_1 \circ \phi_2}$  hold in general, and we conclude:

**Proposition 2.1.** *The construction (2.4) gives rise to a covariant functor  $(u, \phi) \rightarrow (H(u), U_\phi)$  from the category of positive definite functions on sets to the category of complex Hilbert spaces.*

It is significant that if  $X$  is a topological space and  $u : X \times X \rightarrow \mathbb{C}$  is a *continuous* positive definite function, then the associated map  $f : X \rightarrow H(u)$  of (2.4) is also continuous. Indeed, this is immediate from (2.2):

$$\|f(x) - f(y)\|^2 = u(x, x) + u(y, y) - u(x, y) - u(y, x), \quad x, y \in X.$$

The functorial nature of Proposition 2.1 pays immediate dividends:

*Remark 2.2.* [Automorphisms] Every positive definite function

$$u : X \times X \rightarrow \mathbb{C}$$

has an associated group of internal symmetries, namely the group  $G_u$  of all bijections  $\phi : X \rightarrow X$  that preserve  $u$  in the sense that

$$u(\phi(x), \phi(y)) = u(x, y), \quad x, y \in X.$$

Notice that Proposition 2.1 implies that this group of symmetries has a natural unitary representation  $U : G_u \rightarrow \mathcal{B}(H(u))$  associated with it. Indeed, for every  $\phi \in G_u$ , the unitary operator  $U_\phi \in \mathcal{B}(H(u))$  is defined uniquely by

$$U_\phi(f(x)) = f(\phi(x)), \quad x \in X.$$

The properties of this unitary representation of the automorphism group of  $u$  often reflect important features of the environment that produced  $u$ .

**Examples:** There are many examples of positive definite functions; some of the more popular are reproducing kernels associated with domains in  $\mathbb{C}^n$ . Here is another example that is important for quantum physics and happens to be one of my favorites. Let  $Z$  be a (finite or infinite dimensional) Hilbert space and consider the positive definite function  $u : Z \times Z \rightarrow \mathbb{C}$  defined by

$$u(z, w) = e^{\langle z, w \rangle}, \quad z, w \in Z.$$

We will write the Hilbert space  $H(u)$  defined by the construction of Proposition 2.1 as  $e^Z$ , since it can be identified as the symmetric Fock space over the one-particle space  $Z$ . We will not make that identification here, but we do write the natural function (2.4) from  $Z$  to  $e^Z$  as  $f(z) = e^z$ ,  $z \in Z$ .

One finds that the automorphism group of Remark 2.2 is the full unitary group  $\mathcal{U}(Z)$  of  $Z$ . Hence the functorial nature of the preceding construction leads immediately to a (strongly continuous) unitary representation  $\Gamma$  of the unitary group  $\mathcal{U}(Z)$  on the Hilbert space  $e^Z$ . In explicit terms, for  $U \in \mathcal{U}(Z)$ ,  $\Gamma(U)$  is the unique unitary operator of  $e^Z$  that satisfies

$$\Gamma(U)(e^z) = e^{Uz}, \quad U \in \mathcal{U}(Z), \quad z \in Z.$$

The map  $\Gamma$  is called *second quantization* in the physics literature. It has the property that for every one-parameter unitary group  $\{U_t : t \in \mathbb{R}\}$  acting on  $Z$ , there is a corresponding one-parameter unitary group  $\{\Gamma(U_t) : t \in \mathbb{R}\}$  that acts on the “first quantized” Hilbert space  $e^Z$ . Equivalently, for every self adjoint operator  $A$  on  $Z$ , there is a corresponding “second quantized” self adjoint operator  $d\Gamma(A)$  on  $e^Z$  that is uniquely defined by the formula

$$e^{itd\Gamma(A)} = \Gamma(e^{itA}), \quad t \in \mathbb{R},$$

as one sees by applying Stone’s theorem which characterizes the generators of strongly continuous one-parameter unitary groups.

Finally, one can exploit the functorial nature of this construction further to obtain a natural representation of the canonical commutation relations on  $e^Z$ , but we will not pursue that here.

### 3. POSITIVE LINEAR MAPS ON COMMUTATIVE \*-ALGEBRAS

The results of Sections 4 and 5 on subnormal operators, positive operator valued measures and the dilation theory of contractions can all be based on a single dilation theorem for positive linear maps of commutative Banach \*-algebras. That commutative theorem has a direct commutative proof. But

since we require a more general noncommutative dilation theorem in Section 6 that contains it as a special case, we avoid repetition by merely stating the commutative result in this section. What we want to emphasize here is the unexpected appearance of complete positivity even in this commutative context, and the functorial nature of dilation theorems of this kind.

A Banach  $*$ -algebra is a Banach algebra  $\mathcal{A}$  that is endowed with an isometric involution – an antilinear mapping  $a \mapsto a^*$  of  $\mathcal{A}$  into itself that satisfies  $a^{**} = a$ ,  $(ab)^* = b^*a^*$  and  $\|a^*\| = \|a\|$ . In this section we will be concerned with Banach  $*$ -algebras that are *commutative*, and which have a multiplicative unit  $\mathbf{1}$  that satisfies  $\|\mathbf{1}\| = 1$ . The basic properties of Banach  $*$ -algebras and their connections with  $C^*$ -algebras are summarized in Appendix A.

An operator-valued linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  of a Banach  $*$ -algebra is said to be *positive* if  $\phi(a^*a) \geq 0$  for every  $a \in \mathcal{A}$ . The most important fact about operator-valued positive linear maps of commutative algebras is something of a miracle. It asserts that a positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  of a commutative Banach  $*$ -algebra  $\mathcal{A}$  is *completely* positive in the following sense: For every  $n$ -tuple  $a_1, \dots, a_n$  of elements of  $\mathcal{A}$ , the  $n \times n$  operator matrix  $(\phi(a_i^*a_j))$  is positive in the natural sense that for every  $n$ -tuple of vectors  $\xi_1, \dots, \xi_n \in H$ , one has

$$(3.1) \quad \sum_{i,j=1}^n \langle \phi(a_i^*a_j) \xi_j, \xi_i \rangle \geq 0.$$

Notice that the hypothesis  $\phi(a^*a) \geq 0$  is the content of these inequalities for the special case  $n = 1$ . This result for commutative  $C^*$ -algebras  $\mathcal{A}$  is due to Stinespring (see Theorem 4 of [Sti55]), and the proof of (3.1) can be based on that result combined with the properties of the completion map  $\iota : \mathcal{A} \rightarrow C^*(\mathcal{A})$  that carries a commutative Banach  $*$ -algebra  $\mathcal{A}$  to its enveloping  $C^*$ -algebra  $C^*(\mathcal{A}) \cong C(X)$  (see Remark A.3 of Appendix A).

The notion of complete positivity properly belongs to the noncommutative world. We will return to it in Section 6 where we will prove a general result (Theorem 6.1) which, when combined with (3.1), implies the following assertion about positive linear maps of commutative  $*$ -algebras.

**Scholium A:** *Let  $\mathcal{A}$  be a commutative Banach  $*$ -algebra with unit and let  $H$  be a Hilbert space. For every operator-valued linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  satisfying  $\phi(a^*a) \geq 0$ ,  $a \in \mathcal{A}$ , there is a pair  $(V, \pi)$  consisting of a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(K)$  of  $\mathcal{A}$  on another Hilbert space  $K$  and a linear operator  $V \in \mathcal{B}(H, K)$  such that*

$$(3.2) \quad \phi(a) = V^* \pi(a) V, \quad a \in \mathcal{A}.$$

*Moreover,  $\phi$  is necessarily bounded, its norm is given by*

$$(3.3) \quad \sup_{\|a\| \leq 1} \|\phi(a)\| = \|\phi(\mathbf{1})\| = \|V\|^2,$$

*and  $V$  can be taken to be an isometry when  $\phi(\mathbf{1}) = \mathbf{1}$ .*

*Remark 3.1* (Minimality and uniqueness of dilation pairs). Fix  $\mathcal{A}$  as above. By a *dilation pair* for  $\mathcal{A}$  we mean a pair  $(V, \pi)$  consisting of a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(K)$  and a bounded linear map  $V : H \rightarrow K$  from some other Hilbert space  $H$  into the space  $K$  on which  $\pi$  acts. A dilation pair  $(V, \pi)$  is said to be *minimal* if the set of vectors  $\{\pi(a)V\xi : a \in \mathcal{A}, \xi \in H\}$  has  $K$  as its closed linear span. By replacing  $K$  with an appropriate subspace and  $\pi$  with an appropriate subrepresentation, we can obviously replace every such pair with a minimal one. Moreover, the representation associated with a minimal pair must be nondegenerate, and therefore  $\pi(\mathbf{1}) = \mathbf{1}_K$ .

Note that every dilation pair  $(V, \pi)$  gives rise to a positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  that is defined by the formula (3.2), and we say that  $(V, \pi)$  is a dilation pair *for*  $\phi$ . A positive map  $\phi$  has many dilation pairs associated with it, but the minimal ones are equivalent in the following sense: If  $(V_1, \pi_1)$  and  $(V_2, \pi_2)$  are two *minimal* dilation pairs for  $\phi$  then there is a unique unitary operator  $W : K_1 \rightarrow K_2$  such that

$$(3.4) \quad WV_1 = V_2, \quad \text{and} \quad W\pi_1(a) = \pi_2(a)W, \quad a \in \mathcal{A}.$$

The proof amounts to little more than checking inner products on the two generating sets  $\pi_1(\mathcal{A})V_1H \subseteq K_1$  and  $\pi_2(\mathcal{A})V_2H \subseteq K_2$  and noting that

$$\begin{aligned} \langle \pi_2(a)V_2\xi, \pi_2(b)V_2\eta \rangle &= \langle \pi_2(b^*a)V_2\xi, \eta \rangle = \langle \phi(b^*a)\xi, \eta \rangle \\ &= \langle \pi_1(a)V_1\xi, \pi_1(b)V_1\eta \rangle, \end{aligned}$$

for  $a, b \in \mathcal{A}$  and  $\xi, \eta \in H$ .

Finally, note that in cases where  $\phi(\mathbf{1}) = \mathbf{1}$ , the operator  $V$  of a minimal pair  $(V, \pi)$  is an isometry, so by making an obvious identification we can replace  $(V, \pi)$  with an equivalent one in which  $V$  is the inclusion map of  $H$  into a larger Hilbert space  $\iota : H \subseteq K$  and  $\pi$  is a representation of  $\mathcal{A}$  on  $K$ . After these identifications, (3.2) reduces to the more traditional assertion

$$(3.5) \quad \phi(a) = P_H\pi(a) \upharpoonright_H, \quad a \in \mathcal{A}.$$

*Remark 3.2* (Functoriality). It is a worthwhile exercise to think carefully about what a dilation actually *is*, and the way to do that is to think in categorical terms. Fix a commutative Banach  $*$ -algebra  $\mathcal{A}$  with unit  $\mathbf{1}$ . Operator-valued positive linear maps of  $\mathcal{A}$  are the objects of a category, in which a homomorphism from  $\phi_1 : \mathcal{A} \rightarrow \mathcal{B}(H_1)$  to  $\phi_2 : \mathcal{A} \rightarrow \mathcal{B}(H_2)$  is defined as a unitary operator  $U : H_1 \rightarrow H_2$  satisfying  $U\phi_1(a) = \phi_2(a)U$  for all  $a \in \mathcal{A}$ ; equivalently,  $U$  should implement a unitary equivalence of positive linear maps of  $\mathcal{A}$ . Thus the positive linear maps of  $\mathcal{A}$  can be viewed as a groupoid – a category in which every arrow is invertible.

There is a corresponding groupoid whose objects are minimal dilation pairs  $(V, \pi)$ . Homomorphisms of dilation pairs  $(V_1, \pi_1) \rightarrow (V_2, \pi_2)$  (here  $\pi_j$  is a representation of  $\mathcal{A}$  on  $K_j$  and  $V_j$  is an operator in  $\mathcal{B}(H_j, K_j)$ ) are defined as unitary operators  $W : K_1 \rightarrow K_2$  that satisfy

$$W\pi_1(a) = \pi_2(a)W, \quad a \in \mathcal{A}, \quad \text{and} \quad WV_1 = V_2.$$

The “set” of all dilation pairs for a fixed positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  is a subgroupoid, and we have already seen in Remark 3.1 that its elements are all isomorphic. But here we are mainly concerned with how the dilation functor treats arrows between different positive linear maps.

A functor is the end product of a *construction*. In order to describe how the dilation functor acts on arrows, we need more information than the statement of Scholium A contains, namely the following: *There is a construction which starts with a positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  and generates a particular dilation pair  $(V, \pi)_\phi$  from that data.* Scholium A asserts that such dilation pairs exist for every  $\phi$ , but since the proof is missing, we have not seen the construction. Later on, however, we will show how to construct a particular dilation pair  $(V, \pi)_\phi$  from a completely positive map  $\phi$  when we prove Stinespring’s theorem in section 6. That construction is analogous to the construction underlying (2.4), which exhibits an explicit function  $f : X \rightarrow H(u)$  that arises from the construction of the Hilbert space  $H(u)$ , starting with a positive definite function  $u$ . In order to continue the current discussion, we ask the reader to assume the result of the construction of Theorem 6.1, namely that we are somehow given a *particular* dilation pair  $(V, \pi)_\phi$  for every positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$ .

That puts us in position to describe how the dilation functor acts on arrows. Given two positive linear maps  $\phi_j : \mathcal{A} \rightarrow \mathcal{B}(H_j)$ ,  $j = 1, 2$ , let  $U : H_1 \rightarrow H_2$  be a unitary operator satisfying  $U\phi_1(a) = \phi_2(a)U$  for  $a \in \mathcal{A}$ . Let  $(V_1, \pi_1)$  and  $(V_2, \pi_2)$  be the dilation pairs that have been constructed from  $\phi_1$  and  $\phi_2$  respectively. Notice that since  $U^*\phi_2(a)U = \phi_1(a)$  for  $a \in \mathcal{A}$ , it follows that  $(V_2U, \pi_2)$  is a second minimal dilation pair for  $\phi_1$ . By (3.4), there is a unique unitary operator  $\tilde{U} : K_1 \rightarrow K_2$  that satisfies

$$\tilde{U}V_1 = V_2U, \quad \text{and} \quad \tilde{U}\pi_1(a) = \pi_2(a)\tilde{U}, \quad a \in \mathcal{A}.$$

One can now check that the association  $\phi, U \rightarrow (V, \pi)_\phi, \tilde{U}$  defines a covariant functor from the groupoid of operator-valued positive linear maps of  $\mathcal{A}$  to the groupoid of minimal dilation pairs for  $\mathcal{A}$ .

#### 4. SUBNORMALITY

An operator  $A$  on a Hilbert space  $H$  is said to be *subnormal* if it can be extended to a normal operator on a larger Hilbert space. More precisely, there should exist a normal operator  $B$  acting on a Hilbert space  $K \supseteq H$  that leaves  $H$  invariant and restricts to  $A$  on  $H$ . Halmos’ paper [Hal50] introduced the concept, and grew out of his observation that a subnormal operator  $A \in \mathcal{B}(H)$  must satisfy the following system of peculiar inequalities:

$$(4.1) \quad \sum_{i,j=0}^n \langle A^i \xi_j, A^j \xi_i \rangle \geq 0, \quad \forall \xi_0, \xi_1, \dots, \xi_n \in H, \quad n = 0, 1, 2, \dots$$

It is an instructive exercise with inequalities involving  $2 \times 2$  operator matrices to show that the case  $n = 1$  of (4.1) is equivalent to the single operator

inequality  $A^*A \geq AA^*$ , a property called *hyponormality* today. Subnormal operators are certainly hyponormal, but the converse is false even for weighted shifts (see Problem 160 of [Hal67]). Halmos showed that the full set of inequalities (4.1) – together with a second system of necessary inequalities that we do not reproduce here – implies that  $A$  is subnormal. Several years later, his student J. Bram proved that the second system of inequalities follows from the first [Bra55], and simpler proofs of that fact based on semigroup considerations emerged later [Szf77]. Hence the system of inequalities (4.1) is by itself necessary and sufficient for subnormality.

It is not hard to reformulate Halmos' notion of subnormality (for single operators) in a more general way that applies to several operators. Let  $\Sigma$  be a commutative semigroup (written additively) that contains a neutral element 0. By a *representation* of  $\Sigma$  we mean an operator valued function  $s \in \Sigma \mapsto A(s) \in \mathcal{B}(H)$  satisfying  $A(s+t) = A(s)A(t)$  and  $A(0) = \mathbf{1}$ . Notice that we make no assumption on the norms  $\|A(s)\|$  as  $s$  varies over  $\Sigma$ . For example, a commuting set  $A_1, \dots, A_d$  of operators on a Hilbert space  $H$  gives rise to a representation of the  $d$ -dimensional additive semigroup

$$\Sigma = \{(n_1, \dots, n_d) : n_1 \geq 0, \dots, n_d \geq 0\}$$

by way of

$$A(n_1, \dots, n_d) = A_1^{n_1} \cdots A_d^{n_d}, \quad (n_1, \dots, n_d) \in \Sigma.$$

In general, a representation  $A : \Sigma \rightarrow \mathcal{B}(H)$  is said to be *subnormal* if there is a Hilbert space  $K \supseteq H$  and a representation  $B : \Sigma \rightarrow \mathcal{B}(K)$  consisting of normal operators such that each  $B(s)$  leaves  $H$  invariant and

$$B(s) \upharpoonright_H = A(s), \quad s \in \Sigma.$$

We now apply Scholium A to prove a general statement about commutative operator semigroups that contains the Halmos-Bram characterization of subnormal operators, as well as higher dimensional variations of it that apply to semigroups generated by a finite or even infinite number of mutually commuting operators.

**Theorem 4.1.** *Let  $\Sigma$  be a commutative semigroup with 0. A representation  $A : \Sigma \rightarrow \mathcal{B}(H)$  is subnormal iff for every  $n \geq 1$ , every  $s_1, \dots, s_n \in \Sigma$  and every  $\xi_1, \dots, \xi_n \in H$ , one has*

$$(4.2) \quad \sum_{i,j=1}^n \langle A(s_i)\xi_j, A(s_j)\xi_i \rangle \geq 0.$$

*Proof.* The proof that the system of inequalities (4.2) is necessary for subnormality is straightforward, and we omit it. Here we outline a proof of the converse, describing all essential steps in the construction but leaving routine calculations for the reader. We shall make use of the hypothesis (4.2) in the following form: For every function  $s \in \Sigma \mapsto \xi(s) \in H$  such that



$\xi(s)$  vanishes for all but a finite number of  $s \in \Sigma$ , one has

$$(4.3) \quad \sum_{s,t \in \Sigma} \langle A(s)\xi(t), A(t)\xi(s) \rangle \geq 0.$$

We first construct an appropriate commutative Banach  $*$ -algebra. Note that the direct sum of semigroups  $\Sigma \oplus \Sigma$  is a commutative semigroup with zero element  $(0,0)$ , but unlike  $\Sigma$  it has a natural involution  $x \mapsto x^*$  defined by  $(s,t)^* = (t,s)$ ,  $s,t \in \Sigma$ . We will also make use of a weight function  $w : \Sigma \oplus \Sigma \rightarrow [1, \infty)$  defined as follows:

$$w(s,t) = \max(\|A(s)\| \cdot \|A(t)\|, 1), \quad s,t \in \Sigma.$$

Straightforward verification (using  $\|A(s+t)\| \leq \|A(s)\| \cdot \|A(t)\|$ ) establishes the properties

$$1 \leq w(x+y) \leq w(x)w(y), \quad w(x^*) = w(x), \quad x,y \in \Sigma \oplus \Sigma.$$

Note too that  $w((0,0)) = 1$  because  $A(0) = \mathbf{1}$ . Consider the Banach space  $\mathcal{A}$  of all functions  $f : \Sigma \oplus \Sigma \rightarrow \mathbb{C}$  having finite weighted  $\ell^1$ -norm

$$(4.4) \quad \|f\| = \sum_{x \in \Sigma \oplus \Sigma} |f(x)| \cdot w(x) < \infty.$$

Since  $w \geq 1$ , the norm on  $\mathcal{A}$  dominates the ordinary  $\ell^1$  norm, so that every function in  $\mathcal{A}$  belongs to  $\ell^1(\Sigma \oplus \Sigma)$ . Ordinary convolution of functions defined on commutative semigroups

$$(f * g)(z) = \sum_{\{x,y \in \Sigma \oplus \Sigma : x+y=z\}} f(x)g(y), \quad z \in \Sigma \oplus \Sigma$$

defines an associative commutative multiplication in  $\ell^1(\Sigma \oplus \Sigma)$ , and it is easy to check that the above properties of the weight function  $w$  imply that with respect to convolution and the involution  $f^*(s,t) = \bar{f}(t,s)$ ,  $\mathcal{A}$  becomes a commutative Banach  $*$ -algebra with normalized unit  $\delta_{(0,0)}$ .

We now use the semigroup  $A(\cdot)$  to construct a linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$ :

$$\phi(f) = \sum_{(s,t) \in \Sigma \oplus \Sigma} f(s,t) A(s)^* A(t).$$

Note that  $\|\phi(f)\| \leq \|f\|$  because of the definition norm of  $f$  in terms of the weight function  $w$ . Obviously  $\phi(\delta_{(s,t)}) = A(s)^* A(t)$  for all  $s,t \in \Sigma$ , and in particular  $\phi(\delta_{(0,0)}) = \mathbf{1}$ . It is also obvious that  $\phi(f^*) = \phi(f)^*$  for  $f \in \mathcal{A}$ .

What is most important for us is that  $\phi$  is a *positive* linear map, namely for every  $f \in \mathcal{A}$  and every vector  $\xi \in H$

$$(4.5) \quad \langle \phi((f^*) * f)\xi, \xi \rangle \geq 0.$$

To deduce this from (4.3), note that since  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  is a bounded linear map and every function in  $\mathcal{A}$  can be norm-approximated by functions which are finitely nonzero, it suffices to verify (4.5) for functions  $f : \Sigma \oplus \Sigma \rightarrow \mathbb{C}$  such that  $f(x) = 0$  for all but a finite number of  $x \in \Sigma \oplus \Sigma$ . But for two

finitely supported functions  $f, g \in \mathcal{A}$  and any function  $H : \Sigma \oplus \Sigma \rightarrow \mathbb{C}$ , the definition of convolution implies that  $f * g$  is finitely supported, and

$$\sum_{z \in \Sigma \oplus \Sigma} (f * g)(z) H(z) = \sum_{x, y \in \Sigma \oplus \Sigma} f(x) g(y) H(x + y).$$

Fixing  $\xi \in H$  and taking  $H(s, t) = \langle A(s)^* A(t) \xi, \xi \rangle = \langle A(t) \xi, A(s) \xi \rangle$ , we conclude from the preceding formula that

$$\begin{aligned} \langle \phi(f * g) \xi, \xi \rangle &= \sum_{s, t, u, v \in \Sigma} f(s, t) g(u, v) \langle A(t + v) \xi, A(s + u) \xi \rangle \\ &= \sum_{s, t, u, v \in \Sigma} f(s, t) g(u, v) \langle A(t) A(v) \xi, A(u) A(s) \xi \rangle. \end{aligned}$$

Thus we can write

$$\begin{aligned} \langle \phi(f^* * f) \xi, \xi \rangle &= \sum_{s, t, u, v \in \Sigma} \bar{f}(t, s) f(u, v) \langle A(t) A(v) \xi, A(u) A(s) \xi \rangle \\ &= \sum_{t, u \in \Sigma} \langle A(t) (\sum_{v \in \Sigma} f(u, v) A(v) \xi), A(u) (\sum_{s \in \Sigma} f(t, s) A(s) \xi) \rangle \\ &= \sum_{t, u \in \Sigma} \langle A(t) \xi(u), A(u) \xi(t) \rangle, \end{aligned}$$

where  $t \in \Sigma \mapsto \xi(t) \in H$  is the vector function

$$\xi(t) = \sum_{s \in \Sigma} f(t, s) A(s) \xi, \quad t \in \Sigma.$$

Notice that the rearrangements of summations carried out in the preceding formula are legitimate because all sums are finite, and in fact the vector function  $t \mapsto \xi(t)$  is itself finitely nonzero. (4.5) now follows from (4.3).

At this point, we can apply Scholium A to find a Hilbert space  $K$  which contains  $H$  and a  $*$ -representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(K)$  such that

$$P_H \pi(f) \upharpoonright_H = \phi(f), \quad f \in \mathcal{A}.$$

Hence the map  $x \mapsto \pi(\delta_x)$  is a  $*$ -preserving representation of the  $*$ -semigroup  $\Sigma \oplus \Sigma$ , which can be further decomposed by way of  $\pi(\delta_{(s, t)}) = B(s)^* B(t)$ , where  $B : \Sigma \rightarrow \mathcal{B}(K)$  is the representation  $B(t) = \pi(\delta(0, t))$ . Since the commutative semigroup of operators  $\{\pi(\delta_x) : x \in \Sigma \oplus \Sigma\}$  is closed under the  $*$ -operation,  $B(\Sigma)$  is a semigroup of mutually commuting normal operators. After taking  $s = 0$  in the formulas  $P_H B(s)^* B(t) \upharpoonright_H = A(s)^* A(t)$ , one finds that  $A(t)$  is the compression of  $B(t)$  to  $H$ . Moreover, since for every  $t \in \Sigma$

$$P_H B(t)^* B(t) \upharpoonright_H = A(t)^* A(t) = P_H B(t)^* P_H B(t) \upharpoonright_H,$$

we have  $P_H B(t)^* (\mathbf{1} - P_H) B(t) P_H = 0$ . Thus we have shown that  $H$  is invariant under  $B(t)$  and the restriction of  $B(t)$  to  $H$  is  $A(t)$ .  $\square$

*Remark 4.2* (Minimality and functoriality). Let  $\Sigma$  be a commutative semigroup with zero. A normal extension  $s \in \Sigma \mapsto B(s) \in \mathcal{B}(K)$  of a representation  $s \in \Sigma \mapsto A(s) \in \mathcal{B}(H)$  on a Hilbert space  $K \supseteq H$  is said to be *minimal*

if the set of vectors  $\{B(t)^*\xi : t \in \Sigma, \xi \in H\}$  has  $K$  as its closed linear span. This corresponds to the notion of minimality described in Section 6. The considerations of Remark 3.1 imply that all minimal dilations are equivalent, and we can speak unambiguously of *the* minimal normal extension of  $A$ . A similar comment applies to the functorial nature of the map which carries subnormal representations of  $\Sigma$  to their minimal normal extensions.

*Remark 4.3* (Norms and flexibility). It is a fact that the minimal normal extension  $B$  of  $A$  satisfies  $\|B(t)\| = \|A(t)\|$  for  $t \in \Sigma$ . The inequality  $\geq$  is obvious since  $A(t)$  is the restriction of  $B(t)$  to an invariant subspace. However, if one attempts to use the obvious norm estimate for representations of Banach  $*$ -algebras (see Appendix A for more detail) to establish the opposite inequality, one finds that the above construction gives only

$$\|B(t)\| = \|\pi(\delta_{(0,t)})\| \leq \|\delta_{(0,t)}\| = w(0,t) = \max(\|A(t)\|, 1),$$

which is not good enough when  $\|A(t)\| < 1$ . On the other hand, we can use the flexibility in the possible norms of  $\mathcal{A}$  to obtain the correct estimate as follows. For each  $\epsilon > 0$ , define a new weight function  $w_\epsilon$  on  $\Sigma \oplus \Sigma$  by

$$w_\epsilon(s,t) = \max(\|A(s)\| \cdot \|A(t)\|, \epsilon), \quad s, t \in \Sigma.$$

If one uses  $w_\epsilon$  in place of  $w$  in the definition (4.4) of the norm on  $\mathcal{A}$ , one obtains another commutative Banach  $*$ -algebra which serves equally well as the original to construct the minimal normal dilation  $B$  of  $A$ , and it has the additional property that  $\|B(t)\| \leq \max(\|A(t)\|, \epsilon)$  for  $t \in \Sigma$ . Since  $\epsilon$  can be arbitrarily small, the desired estimate  $\|B(t)\| \leq \|A(t)\|$  follows. In particular, for every  $t \in \Sigma$  we have  $A(t) = 0 \implies B(t) = 0$ .

## 5. COMMUTATIVE DILATION THEORY

Dilation theory began with two papers of Naimark, written and published somehow during the darkest period of world war II: [Nai43a], [Nai43b]. Naimark's theorem asserts that a countably additive measure  $E : \mathcal{F} \rightarrow \mathcal{B}(H)$  defined on a  $\sigma$ -algebra  $\mathcal{F}$  of subsets of a set  $X$  that takes values in the set of positive operators on a Hilbert space  $H$  and satisfies  $E(X) = \mathbf{1}$  can be expressed in the form

$$E(S) = P_H Q(S) \upharpoonright_H, \quad S \in \mathcal{F},$$

where  $K$  is a Hilbert space containing  $H$  and  $Q : \mathcal{F} \rightarrow \mathcal{B}(K)$  is a spectral measure. A version of Naimark's theorem (for regular Borel measures on topological spaces) can be found on p. 50 of [Pau02]. Positive operator valued measures  $E$  have become fashionable in quantum physics and quantum information theory, where they go by the unpronounceable acronym POVM. It is interesting that the Wikipedia page for projection-operator-valued-measures (<http://en.wikipedia.org/wiki/POVM>) contains more information about Naimark's famous theorem than the Wikipedia page for Naimark himself ([http://en.wikipedia.org/wiki/Mark\\_Naimark](http://en.wikipedia.org/wiki/Mark_Naimark)).

In his subnormality paper [Hal50], Halmos showed that every contraction  $A \in \mathcal{B}(H)$  has a unitary dilation in the sense that there is a unitary operator  $U$  acting on a larger Hilbert space  $K \supseteq H$  such that

$$A = P_H U \upharpoonright_H.$$

Sz.-Nagy extended that in a most significant way [SN53] by showing that every contraction has a unitary *power* dilation, and the latter result ultimately became the cornerstone for an effective model theory for Hilbert space contractions [SNF70]. Today, these results belong to the toolkit of every operator theorist, and can be found in many textbooks. In this section we merely state Sz.-Nagy's theorem and sketch a proof that is in the spirit of the preceding discussion.

**Theorem 5.1.** *Let  $A \in \mathcal{B}(H)$  be an operator satisfying  $\|A\| \leq 1$ . Then there is a unitary operator  $U$  acting on a Hilbert space  $K$  containing  $H$  such that*

$$(5.1) \quad A^n = P_H U^n \upharpoonright_H, \quad n = 0, 1, 2, \dots$$

*If  $U$  is minimal in the sense that  $K$  is the closed linear span of  $\cup_{n \in \mathbb{Z}} U^n H$ , then it is uniquely determined up to a natural unitary equivalence.*

*Sketch of proof.* Consider the commutative Banach  $*$ -algebra  $\mathcal{A} = \ell^1(\mathbb{Z})$ , with multiplication and involution given by

$$(f * g)(n) = \sum_{k=-\infty}^{+\infty} f(k)g(n-k), \quad f^*(n) = \bar{f}(-n), \quad n \in \mathbb{Z},$$

and normalized unit  $\mathbf{1} = \delta_0$ . Define  $A(n) = A^n$  if  $n \geq 0$  and  $A(n) = A^{*|n|}$  if  $n < 0$ . Since  $\|A(n)\| \leq 1$  for every  $n$ , we can define a linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  in the obvious way

$$\phi(f) = \sum_{n=-\infty}^{+\infty} f(n)A(n), \quad f \in \mathcal{A}.$$

It is obvious that  $\|\phi(f)\| \leq \|f\|$ ,  $f \in \mathcal{A}$ , but not at all obvious that  $\phi$  is a positive linear map. However, there is a standard method for showing that for every  $\xi \in H$ , the sequence of complex numbers  $a_n = \langle A(n)\xi, \xi \rangle$ ,  $n \in \mathbb{Z}$ , is of positive type in the sense that for every finitely nonzero sequence of complex numbers  $\lambda_n$ ,  $n \in \mathbb{Z}$ , one has  $\sum_{n \in \mathbb{Z}} a_{n-m} \lambda_n \bar{\lambda}_m \geq 0$ ; for example, see p. 36 of [Pau02]. By approximating  $f \in \mathcal{A}$  in the norm of  $\mathcal{A}$  with finitely nonzero functions and using

$$\langle \phi((f^*) * f)\xi, \xi \rangle = \sum_{m,n=-\infty}^{+\infty} \langle A(n-m)\xi, \xi \rangle f(n) \bar{f}(m),$$

it follows that  $\langle \phi(f)\xi, \xi \rangle \geq 0$ , and we may conclude that  $\phi$  is a positive linear map of  $\mathcal{A}$  to  $\mathcal{B}(H)$  satisfying  $\phi(\delta_0) = \mathbf{1}$ .

Scholium A implies that there is a  $*$ -representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(K)$  of  $\mathcal{A}$  on a larger Hilbert space  $K$  such that  $\pi(f)$  compresses to  $\phi(f)$  for  $f \in \mathcal{A}$ . Finally, since the enveloping  $C^*$ -algebra of  $\mathcal{A} = \ell^1(\mathbb{Z})$  is the commutative  $C^*$ -algebra  $C(\mathbb{T})$ , the representation  $\pi$  promotes to a representation  $\tilde{\pi} : C(\mathbb{T}) \rightarrow \mathcal{B}(K)$  (see Appendix A). Taking  $z \in C(\mathbb{T})$  to be the coordinate variable, we obtain a unitary operator  $U \in \mathcal{B}(K)$  by way of  $U = \tilde{\pi}(z)$ , and formula (5.1) follows. We omit the proof of the last sentence.  $\square$

No operator theorist can resist repeating the elegant proof of von Neumann's inequality that flows from Theorem 5.1. von Neumann's inequality [vN51] asserts that for every operator  $A \in \mathcal{B}(H)$  satisfying  $\|A\| \leq 1$ , one has

$$(5.2) \quad \|f(A)\| \leq \sup_{|z| \leq 1} |f(z)|$$

for every polynomial  $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ . von Neumann's original proof was difficult, involving calculations with Möbius transformations and Blaschke products. Letting  $U \in \mathcal{B}(K)$  be a unitary power dilation of  $A$  satisfying (5.1), one has  $f(A) = P_H f(U) \upharpoonright_H$ , for every polynomial  $f$ , hence

$$\|f(A)\| \leq \|f(U)\| = \sup_{z \in \sigma(U)} |f(z)| \leq \sup_{|z|=1} |f(z)|.$$

## 6. COMPLETELY POSITIVITY AND STINESPRING'S THEOREM

While one can argue that the GNS construction for states of  $C^*$ -algebras is a dilation theorem, it is probably best thought of as an application of the general method of associating a Hilbert space with a positive definite function as described in Section 2. Dilation theory *proper* went noncommutative in 1955 with the publication of a theorem of Stinespring [Sti55]. Stinespring once told me that his original motivation was simply to find a common generalization of Naimark's commutative result that a positive operator valued measure can be dilated to a spectral measure and the GNS construction for states of (noncommutative)  $C^*$ -algebras. The theorem that emerged went well beyond that, and today has become a pillar upon which significant parts of operator theory and operator algebras rest. The fundamental idea underlying the result was that of a completely positive linear map.

The notion of positive linear functional or positive linear map is best thought of in a purely *algebraic* way. More specifically, let  $\mathcal{A}$  be a  $*$ -algebra, namely a complex algebra endowed with an antilinear mapping  $a \mapsto a^*$  satisfying  $(ab)^* = b^* a^*$  and  $a^{**} = a$  for all  $a, b \in \mathcal{A}$ . An operator-valued linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  (and in particular a complex-valued linear functional  $\phi : \mathcal{A} \rightarrow \mathbb{C}$ ) is called *positive* if it satisfies

$$(6.1) \quad \phi(a^* a) \geq 0, \quad a \in \mathcal{A}.$$

One can promote this notion of positivity to matrix algebras over  $\mathcal{A}$ . For every  $n = 1, 2, \dots$ , the algebra  $M_n(\mathcal{A})$  of  $n \times n$  matrices over  $\mathcal{A}$  has a natural involution, in which the adjoint of an  $n \times n$  matrix is defined as the transposed matrix of adjoints  $(a_{ij})^* = (a_{ji}^*)$ ,  $1 \leq i, j \leq n$ ,  $a_{ij} \in \mathcal{A}$ .

Fixing  $n \geq 1$ , a linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  induces a linear map  $\phi_n$  from  $M_n(\mathcal{A})$  to  $n \times n$  operator matrices  $(\phi(a_{ij}))$  which, after making the obvious identifications, can be viewed as a linear map of  $M_n(\mathcal{A})$  to operators on the direct sum of  $n$  copies of  $H$ . It makes good sense to say that  $\phi_n$  is a positive linear map, and the original map  $\phi$  is called *completely positive* if each  $\phi_n$  is a positive linear map. More explicitly, complete positivity at level  $n$  requires that (6.1) should hold for  $n \times n$  matrices: For every  $n \times n$  matrix  $A = (a_{ij})$  with entries in  $\mathcal{A}$  and every  $n$ -tuple of vectors  $\xi_1, \dots, \xi_n \in H$ , the  $n \times n$  matrix  $B = (b_{ij})$  defined by  $B = A^*A$  satisfies

$$\sum_{i,j=1}^n \langle \phi(b_{ij})\xi_j, \xi_i \rangle = \sum_{i,j,k=1}^n \langle \phi(a_{ki}^*a_{kj})\xi_j, \xi_i \rangle \geq 0.$$

Note that this system of inequalities reduces to a somewhat simpler-looking system of inequalities (3.1) that we have already encountered in Section 3.

If  $\mathcal{A}$  happens to be a  $C^*$ -algebra, then the elements  $x \in \mathcal{A}$  that can be represented in the form  $x = a^*a$  for some  $a \in \mathcal{A}$  are precisely the self adjoint operators  $x$  having nonnegative spectrum. Since  $M_n(\mathcal{A})$  is also a  $C^*$ -algebra in a unique way for every  $n \geq 1$ , completely positive linear maps of  $C^*$ -algebras have a very useful spectral characterization: they should map self adjoint  $n \times n$  operator matrices with nonnegative spectrum to self adjoint operators with nonnegative spectrum. Unfortunately, this spectral characterization breaks down completely for positive linear maps of more general Banach  $*$ -algebras, and in that more general context one must always refer back to positivity as it is expressed in (6.1).

Stinespring's original result was formulated in terms of operator maps defined on  $C^*$ -algebras. We want to reformulate it somewhat into the more flexible context of linear maps of Banach  $*$ -algebras.

**Theorem 6.1.** *Let  $\mathcal{A}$  be a Banach  $*$ -algebra with normalized unit and let  $H$  be a Hilbert space. For every completely positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  there is a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(K)$  of  $\mathcal{A}$  on another Hilbert space  $K$  and a bounded linear map  $V : H \rightarrow K$  such that*

$$(6.2) \quad \phi(a) = V^*\pi(a)V, \quad a \in \mathcal{A}.$$

Moreover, the norm of the linking operator  $V$  is given by  $\|V\|^2 = \|\phi(\mathbf{1})\|$ .

We have omitted the statement and straightforward proof of the converse, namely that every linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  of the form (6.2) must be completely positive, in order to properly emphasize the *construction* of the dilation from the basic properties of a completely positive map.

*Proof.* The underlying construction is identical with the original [Sti55], but a particular estimate requires care in the context of Banach  $*$ -algebras, and we will make that explicit. Consider the tensor product of complex vector

spaces  $\mathcal{A} \otimes H$ , and the sesquilinear form  $\langle \cdot, \cdot \rangle$  defined on it by setting

$$\left\langle \sum_{j=1}^m a_j \otimes \xi_j, \sum_{k=1}^n b_k \otimes \eta_k \right\rangle = \sum_{j,k=1}^{m,n} \langle \phi(b_k^* a_j) \xi_j, \eta_k \rangle.$$

The fact that  $\phi$  is completely positive implies that  $\langle \zeta, \zeta \rangle \geq 0$  for every  $\zeta \in \mathcal{A} \otimes H$ . Letting  $\mathcal{N} = \{\zeta \in \mathcal{A} \otimes H : \langle \zeta, \zeta \rangle = 0\}$ , the Schwarz inequality implies that  $\mathcal{N}$  is a linear subspace and that the sesquilinear form can be promoted to an inner product on the quotient  $K_0 = (\mathcal{A} \otimes H)/\mathcal{N}$ . Let  $K$  be the completion of the resulting inner product space.

Each  $a \in \mathcal{A}$  gives rise to a left multiplication operator  $\pi(a)$  acting on  $\mathcal{A} \otimes H$ , defined uniquely by  $\pi(a)(b \otimes \xi) = ab \otimes \xi$  for  $b \in \mathcal{A}$  and  $\xi \in H$ . The critical estimate that we require is

$$(6.3) \quad \langle \pi(a)\zeta, \pi(a)\zeta \rangle \leq \|a\|^2 \langle \zeta, \zeta \rangle, \quad a \in \mathcal{A}, \quad \zeta \in \mathcal{A} \otimes H,$$

and it is proved as follows. Writing  $\zeta = a_1 \otimes \xi_1 + \cdots + a_n \otimes \xi_n$ , we find that

$$\begin{aligned} \langle \pi(a)\zeta, \pi(a)\zeta \rangle &= \sum_{j,k=1}^n \langle ab_j \otimes \xi_j, ab_k \otimes \xi_k \rangle = \sum_{j,k=1}^n \langle \phi(b_k^* a^* ab_j) \xi_j, \xi_k \rangle \\ &= \sum_{j,k=1}^n \langle a^* ab_j \otimes \xi_j, b_k \otimes \xi_k \rangle = \langle \pi(a^* a)\zeta, \zeta \rangle. \end{aligned}$$

This formula implies that the linear functional  $\rho(a) = \langle \pi(a)\zeta, \zeta \rangle$  satisfies  $\rho(a^* a) = \langle \pi(a)\zeta, \pi(a)\zeta \rangle \geq 0$ . Proposition A.1 of the appendix implies

$$\rho(a^* a) \leq \rho(\mathbf{1}) \|a^* a\| \leq \|\zeta\|^2 \|a\|^2,$$

and (6.3) follows.

It is obvious that  $\pi(ab) = \pi(a)\pi(b)$  and that  $\pi(\mathbf{1})$  is the identity operator. Moreover, as in the argument above, we have  $\langle \pi(a)\eta, \zeta \rangle = \langle \eta, \pi(a^*)\zeta \rangle$  for all  $a \in \mathcal{A}$  and  $\eta, \zeta \in \mathcal{A} \otimes H$ . Finally, (6.3) implies that  $\pi(a)\mathcal{N} \subseteq \mathcal{N}$ , so that each operator  $\pi(a)$ ,  $a \in \mathcal{A}$ , promotes naturally to a linear operator on the quotient  $K_0 = (\mathcal{A} \otimes H)/\mathcal{N}$ . Together with (6.3), these formulas imply that  $\pi$  gives rise to a  $*$  representation of  $\mathcal{A}$  as bounded operators on  $K_0$  which extends uniquely to a representation of  $\mathcal{A}$  on the completion  $K$  of  $K_0$ , which we denote by the same letter  $\pi$ .

It remains only to discuss the connecting operator  $V$ , which is defined by  $V\xi = \mathbf{1} \otimes \xi + \mathcal{N}$ ,  $\xi \in H$ . One finds that  $\pi(a)V\xi = a \otimes \xi + \mathcal{N}$ , from which it follows that

$$\langle \pi(a)V\xi, V\eta \rangle = \langle a \otimes \xi + \mathcal{N}, \mathbf{1} \otimes \eta + \mathcal{N} \rangle = \langle \phi(a)\xi, \eta \rangle, \quad \xi, \eta \in H.$$

Taking  $a = \mathbf{1}$ , we infer that  $\|V\|^2 = \|V^*V\| = \|\phi(\mathbf{1})\|$ , and at that point the preceding formula implies  $\phi(a) = V^*\pi(a)V$ ,  $a \in \mathcal{A}$ .  $\square$

## 7. OPERATOR SPACES, OPERATOR SYSTEMS AND EXTENSIONS

In this section we discuss the basic features of operator spaces and their matrix hierarchies, giving only the briefest of overviews. The interested reader is referred to one of the monographs [BLM04], [ER00], [Pau86] for more about this developing area of noncommutative analysis.

Complex Banach spaces are the objects of a category whose maps are contractions – linear operators of norm  $\leq 1$ . The isomorphisms of this category are surjective isometries. A *function space* is a norm-closed linear subspace of some  $C(X)$  – the space of (complex-valued) continuous functions on a compact Hausdorff space  $X$ , endowed with the sup norm. All students of analysis know that every Banach space  $E$  is isometrically isomorphic to a function space. Indeed, the Hahn-Banach theorem implies that the natural map  $\iota : E \rightarrow E''$  of  $E$  into its double dual has the stated property after one views elements of  $E''$  as continuous functions on the weak\*-compact unit ball  $X$  of  $E'$ . In this way the study of Banach spaces can be reduced to the study of function spaces, and that fact is occasionally useful.

An *operator space* is a norm-closed linear subspace  $\mathcal{E}$  of the algebra  $\mathcal{B}(H)$  of all bounded operators on a Hilbert space  $H$ . Such an  $\mathcal{E}$  is itself a Banach space, and is therefore isometrically isomorphic to a function space. However, the key fact about operator spaces is that they determine an entire hierarchy of operator spaces, one for every  $n = 1, 2, \dots$ . Indeed, for every  $n$ , the space  $M_n(\mathcal{E})$  of all  $n \times n$  matrices over  $\mathcal{E}$  is naturally an operator subspace of  $\mathcal{B}(n \cdot H)$ ,  $n \cdot H$  denoting the direct sum of  $n$  copies of  $H$ . Most significantly, a linear map of operator spaces  $\phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  determines a sequence of linear maps  $\phi_n : M_n(\mathcal{E}_1) \rightarrow M_n(\mathcal{E}_2)$ , where  $\phi_n$  is the linear map obtained by applying  $\phi$  element-by-element to an  $n \times n$  matrix over  $\mathcal{E}_1$ . One says that  $\phi$  is a *complete isometry* or a *complete contraction* if every  $\phi_n$  is, respectively, an isometry or a contraction. There is a corresponding notion of *complete boundedness* that will not concern us here.

Operator spaces can be viewed as the objects of a category whose maps are *complete* contractions. The isomorphisms of this category are complete isometries, and one is led to seek properties of operator spaces that are invariant under this refined notion of isomorphism. Like Shiva, a given Banach space acquires many inequivalent likenesses as an operator space. And in operator space theory one pays attention to what happens at every level of the hierarchy. The result is a significant and fundamentally noncommutative refinement of classical Banach space theory.

For example, since an operator space  $\mathcal{E} \subseteq \mathcal{B}(H)$  is an “ordinary” Banach space, it can be represented as a function system  $\iota : \mathcal{E} \rightarrow C(X)$  as in the opening paragraphs of this section. If we form the hierarchy of  $C^*$ -algebras  $M_n(C(X))$ ,  $n = 1, 2, \dots$ , then we obtain a sequence of embeddings

$$\iota_n : M_n(\mathcal{E}) \rightarrow M_n(C(X)), \quad n = 1, 2, \dots$$



Note that the  $C^*$ -algebra  $M_n(C(X))$  is basically the  $C^*$ -algebra of all matrix-valued continuous functions  $F : X \rightarrow M_n(\mathbb{C})$ , with norm

$$\|F\| = \sup_{x \in X} \|F(x)\|, \quad F \in M_n(C(X)).$$

While the map  $\iota$  is surely an isometry at the first level  $n = 1$ , it may or may not be a complete isometry; indeed for the more interesting examples of operator spaces it is not. Ultimately, the difference between these two categories can be traced to the noncommutativity of operator multiplication, and for that reason some analysts like to think of operator space theory as the “quantized” reformulation of functional analysis.

Finally, one can think of operator spaces somewhat more flexibly as norm-closed linear subspaces  $\mathcal{E}$  of unital (or even nonunital)  $C^*$ -algebras  $\mathcal{A}$ . That is because the hierarchy of  $C^*$ -algebras  $M_n(\mathcal{A})$  is well defined independently of any particular faithful realization  $\mathcal{A}$  as a  $C^*$ -subalgebra of  $\mathcal{B}(H)$ .

One can import the notion of *order* into the theory of operator spaces in a natural way. A *function system* is a function space  $E \subseteq C(X)$  with the property that  $E$  is closed under complex conjugation and contains the constants. One sometimes assumes that  $E$  separates points of  $X$  but we do not. Correspondingly, an *operator system* is a self-adjoint operator space  $\mathcal{E} \subseteq \mathcal{B}(H)$  that contains the identity operator  $\mathbf{1}$ . The natural notion of order between self adjoint operators, namely  $A \leq B \iff B - A$  is a positive operator, has meaning in any operator system  $\mathcal{E}$ , and in fact every operator system is linearly spanned by its positive operators. Every member  $M_n(\mathcal{E})$  of the matrix hierarchy over an operator system  $\mathcal{E}$  is an operator system, so that it makes sense to speak of completely positive maps from one operator system to another.

Krein’s version of the Hahn-Banach theorem implies that a positive linear functional defined on an operator system  $\mathcal{E}$  in a  $C^*$ -algebra  $\mathcal{A}$  can be extended to a positive linear functional on all of  $\mathcal{A}$ . It is significant that this extension theorem fails in general for operator-valued positive linear maps. Fortunately, the following result of [Arv69] provides an effective noncommutative counterpart of Krein’s order-theoretic Hahn-Banach theorem:

**Theorem 7.1.** *Let  $\mathcal{E} \subseteq \mathcal{A}$  be an operator system in a unital  $C^*$ -algebra. Then every operator-valued completely positive linear map  $\phi : \mathcal{E} \rightarrow \mathcal{B}(H)$  can be extended to a completely positive linear map of  $\mathcal{A}$  into  $\mathcal{B}(H)$ .*

There is a variant of 7.1 that looks more like the original Hahn-Banach theorem. Let  $\mathcal{E} \subseteq \mathcal{A}$  be an operator space in a  $C^*$ -algebra  $\mathcal{A}$ . Then every operator-valued *complete* contraction  $\phi : \mathcal{E} \rightarrow \mathcal{B}(H)$  can be extended to a completely contractive linear map of  $\mathcal{A}$  to  $\mathcal{B}(H)$ . While the latter extension theorem emerged more than a decade after Theorem 7.1 (with a different and longer proof [Wit81], [Wit84]), Vern Paulsen discovered a simple device that enables one to deduce it readily from the earlier result. That construction begins with an operator space  $\mathcal{E} \subseteq \mathcal{A}$  and generates an associated operator

system  $\tilde{\mathcal{E}}$  in the  $2 \times 2$  matrix algebra  $M_2(\mathcal{A})$  over  $\mathcal{A}$  as follows:

$$\tilde{\mathcal{E}} = \left\{ \begin{pmatrix} \lambda \cdot \mathbf{1} & A \\ B^* & \lambda \cdot \mathbf{1} \end{pmatrix} : A, B \in \mathcal{E}, \lambda \in \mathbb{C} \right\}.$$

Given a completely contractive linear map  $\phi : \mathcal{E} \rightarrow \mathcal{B}(H)$ , one can define a linear map  $\tilde{\phi} : \tilde{\mathcal{E}} \rightarrow \mathcal{B}(H \oplus H)$  in a natural way

$$\tilde{\phi} \left( \begin{pmatrix} \lambda \cdot \mathbf{1} & A \\ B^* & \lambda \cdot \mathbf{1} \end{pmatrix} \right) = \begin{pmatrix} \lambda \cdot \mathbf{1} & \phi(A) \\ \phi(B)^* & \lambda \cdot \mathbf{1} \end{pmatrix},$$

and it is not hard to see that  $\tilde{\phi}$  is completely positive (I have reformulated the construction in a minor but equivalent way for simplicity; see Lemma 8.1 of [Pau02] for the original). By Theorem 7.1,  $\tilde{\phi}$  extends to a completely positive linear map of  $M_2(\mathcal{A})$  to  $\mathcal{B}(H \oplus H)$ , and the behavior of that extension on the upper right corner is a completely contractive extension of  $\phi$ .

## 8. SPECTRAL SETS AND HIGHER DIMENSIONAL OPERATOR THEORY

Some aspects of commutative operator theory can be properly understood only when placed in the noncommutative context of the matrix hierarchies of the preceding section. In this section we describe the phenomenon in concrete terms, referring the reader to the literature for technical details.

Let  $A \in \mathcal{B}(H)$  be a Hilbert space operator. If  $f$  is a rational function of a single complex variable that has no poles on the spectrum of  $A$ , then there is an obvious way to define an operator  $f(A) \in \mathcal{B}(H)$ . Now fix a compact subset  $X \subseteq \mathbb{C}$  of the plane that contains the spectrum of  $A$ . The algebra  $R(X)$  of all rational functions whose poles lie in the complement of  $X$  forms a unital subalgebra of  $C(X)$ , and this functional calculus defines a unit-preserving homomorphism  $f \mapsto f(A)$  of  $R(X)$  into  $\mathcal{B}(H)$ . One says that  $X$  is a *spectral set* for  $A$  if this homomorphism has norm 1:

$$(8.1) \quad \|f(A)\| \leq \sup_{z \in X} |f(z)|, \quad f \in R(X).$$

von Neumann's inequality (5.2) asserts that the closed unit disk is a spectral set for every contraction  $A \in \mathcal{B}(H)$ ; indeed, that property is characteristic of contractions. While there is no corresponding characterization of the operators that have a more general set  $X$  as a spectral set, we are still free to consider the class of operators that *do* have  $X$  as a spectral set and ask if there is a generalization of Theorem 5.1 that would apply to them. Specifically, given an operator  $A \in \mathcal{B}(H)$  that has  $X$  as a spectral set, is there a normal operator  $N$  acting on a larger Hilbert space  $K \supseteq H$  such that the spectrum of  $N$  is contained in the boundary  $\partial X$  of  $X$  and

$$(8.2) \quad f(A) = P_H f(N) \upharpoonright_H, \quad f \in R(X)?$$

A result of Foias implies that the answer is yes if the complement of  $X$  is connected, but it is no in general. The reason the answer is no in general is that the hypothesis (8.1) is not strong enough; and that phenomenon originates in the *noncommutative* world. To see how the hypothesis

must be strengthened, let  $N$  be a normal operator with spectrum in  $\partial X$ . The functional calculus for normal operators gives rise to a representation  $\pi : C(\partial X) \rightarrow \mathcal{B}(K)$ ,  $\pi(f) = f(N)$ ,  $f \in C(\partial X)$ . It is easy to see that representations of  $C^*$ -algebras are completely positive and completely contractive linear maps, hence if the formula (8.2) holds then the map  $f \in R(X) \mapsto f(A)$  must be not only be a contraction, it must be a *complete* contraction.

Let us examine the latter assertion in more detail. Fix  $n = 1, 2, \dots$  and let  $M_n(R(X))$  be the algebra of all  $n \times n$  matrices with entries in  $R(X)$ . One can view an element of  $M_n(R(X))$  as a matrix valued rational function

$$F : z \in X \mapsto F(z) = (f_{ij}(z)) \in M_n(\mathbb{C}),$$

whose component functions belong to  $R(X)$ . Notice that we can apply such a matrix valued function to an operator  $A$  that has spectrum in  $X$  to obtain an  $n \times n$  matrix of operators – or equivalently an operator  $F(A) = (f_{ij}(A))$  in  $\mathcal{B}(n \cdot H)$ . The map  $F \in M_n(R(X)) \mapsto F(A) \in \mathcal{B}(n \cdot H)$  is a unit-preserving homomorphism of complex algebras.  $X$  is said to be a *complete* spectral set for an operator  $A \in \mathcal{B}(H)$  if it contains the spectrum of  $A$  and satisfies

$$(8.3) \quad \|F(A)\| \leq \sup_{z \in X} \|F(z)\|, \quad F \in M_n(R(X)), \quad n = 1, 2, \dots$$

Now if there is a normal operator  $N$  with spectrum in  $\partial X$  that relates to  $A$  as in (8.2), then for every  $n = 1, 2, \dots$

$$\|F(A)\| \leq \|F(N)\| \leq \sup_{z \in \partial X} \|F(z)\| = \sup_{z \in X} \|F(z)\|,$$

and we conclude that  $X$  must be a *complete* spectral set for  $A$ .

The following result from [Arv72] implies that complete spectral sets suffice for the existence of normal dilations. It depends in an essential way on the extension theorem (Theorem 7.1) for completely positive maps.

**Theorem 8.1.** *Let  $A \in \mathcal{B}(H)$  be an operator that has a compact set  $X \subseteq \mathbb{C}$  as a complete spectral set. Then there is a normal operator  $N$  on a Hilbert space  $K \supseteq H$  having spectrum in  $\partial X$  such that*

$$f(A) = P_H f(N) \upharpoonright_H, \quad f \in R(X).$$

The unitary power dilation of a contraction is unique up to natural equivalence. That reflects a property of the unit circle  $\mathbb{T}$ : A positive linear map  $\phi : C(\mathbb{T}) \rightarrow \mathcal{B}(H)$  is uniquely determined by its values on the nonnegative powers  $1, z, z^2, \dots$  of the current variable  $z$ . In general, however, positive linear maps of  $C(X)$  are not uniquely determined by their values on subalgebras of  $C(X)$ , with the result that there is no uniqueness assertion to complement the existence assertion of Theorem 8.1 for the dilation theory of complete spectral sets.

On the other hand, there is a “many operators” generalization of Theorem 8.1 that applies to completely contractive unit-preserving homomorphisms of arbitrary function algebras  $A \subseteq C(X)$  that act on compact Hausdorff spaces  $X$ , in which  $\partial X$  is replaced by the Silov boundary of  $X$  relative to  $A$ . The details can be found in Theorem 1.2.2 of [Arv72] and its Corollary.

## 9. COMPLETELY POSITIVE MAPS AND ENDOMORPHISMS

In recent years, certain problems arising in mathematical physics and quantum information theory have led researchers to seek a different kind of dilation theory, one that applies to semigroups of completely positive linear maps that act on von Neumann algebras. In this section, we describe the simplest of these dilation theorems as it applies to the simplest semigroups acting on the simplest of von Neumann algebras. A fuller accounting of these developments together with references to other sources can be found in Chapter 8 of the monograph [Arv03].

Let  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  be a unit-preserving completely positive (UCP) map which is *normal* in the sense that for every normal state  $\rho$  of  $\mathcal{B}(H)$ , the composition  $\rho \circ \phi$  is also a normal state. One can think of the semigroup

$$\{\phi^n : n = 0, 1, 2, \dots\}$$

as representing the discrete time evolution of an irreversible quantum system. What we seek is another Hilbert space  $K$  together with a normal  $*$ -endomorphism  $\alpha : \mathcal{B}(K) \rightarrow \mathcal{B}(K)$  that is in some sense a “power dilation” of  $\phi$ . There are a number of ways one can make that vague idea precise, but only one of them is completely effective. It is described as follows.

Let  $K \supseteq H$  be a Hilbert space that contains  $H$  and suppose we are given a normal  $*$ -endomorphism  $\alpha : \mathcal{B}(K) \rightarrow \mathcal{B}(K)$  that satisfies  $\alpha(\mathbf{1}) = \mathbf{1}$ . We write the projection  $P_H$  of  $K$  on  $H$  simply as  $P$ , and we identify  $\mathcal{B}(H)$  with the corner  $P\mathcal{B}(K)P \subseteq \mathcal{B}(K)$ .  $\alpha$  is said to be a *dilation* of  $\phi$  if

$$(9.1) \quad \phi^n(A) = P\alpha^n(A)P, \quad A \in \mathcal{B}(H) = P\mathcal{B}(K)P, \quad n = 0, 1, 2, \dots$$

Since  $\phi$  is a unit-preserving map of  $\mathcal{B}(H)$ ,  $P = \phi(P) = P\alpha(P)P$ , so that  $\alpha(P) \geq P$ . Hence we obtain an increasing sequence of projections

$$(9.2) \quad P \leq \alpha(P) \leq \alpha^2(P) \leq \dots$$

The limit projection  $P_\infty = \lim_n \alpha^n(P)$  satisfies  $\alpha(P_\infty) = P_\infty$ , hence the compression of  $\alpha$  to the larger corner  $P_\infty\mathcal{B}(K)P_\infty \cong \mathcal{B}(P_\infty K)$  of  $\mathcal{B}(K)$  is a unital  $*$ -endomorphism that is itself a dilation of  $\phi$ . By cutting down if necessary we can assume that the configuration is *proper* in the sense that

$$(9.3) \quad \lim_{n \rightarrow \infty} \alpha^n(P) = \mathbf{1}_K,$$

and in that case the endomorphism  $\alpha$  is said to be a *proper* dilation of  $\phi$ . We have refrained from using the term *minimal* to describe this situation because in the context of semigroups of completely positive maps, the notion of *minimal* dilation is a more subtle one that requires a stronger hypothesis. That hypothesis is discussed in Remark 9.3 below.

*Remark 9.1* (Stinespring’s theorem is not enough). It is by no means obvious that dilations should exist. One might attempt to construct a dilation of the semigroup generated by a single UCP map  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  by applying Stinespring’s theorem to the individual terms of the sequence of powers  $\phi^n$ ,  $n = 0, 1, 2, \dots$ , and then somehow putting the pieces together to obtain the

dilating endomorphism  $\alpha$ . Indeed, Stinespring's theorem provides us with a Hilbert space  $K_n \supseteq H$  and a representation  $\pi_n : \mathcal{B}(H) \rightarrow \mathcal{B}(K_n)$  for every  $n \geq 0$  such that

$$\phi^n(A) = P_H \pi_n(A) \upharpoonright_H, \quad A \in \mathcal{B}(H), \quad n = 0, 1, 2, \dots$$

However, while these formulas certainly inherit a relation to each other by virtue of the semigroup formula  $\phi^{m+n} = \phi^m \circ \phi^n$ ,  $m, n \geq 0$ , if one attempts to exploit these relationships one finds that the relation between  $\pi_m$ ,  $\pi_n$  and  $\pi_{m+n}$  is extremely awkward. Actually, there is no apparent way to assemble the von Neumann algebras  $\pi_n(\mathcal{B}(H))$  into a single von Neumann algebra that plays the role of  $\mathcal{B}(K)$ , on which one can define a single endomorphism  $\alpha$  that converts these formulas into the single formula (9.1). Briefly put, *Stinespring's theorem does not apply to semigroups*.

These observations suggest that the problem of constructing dilations in this context should require an entirely new method, and it does. The proper result for normal UCP maps acting on  $\mathcal{B}(H)$  was discovered by Bhat and Parthasarathy [BP94], building on earlier work of Parthasarathy [Par91] that was set in the context of quantum probability theory. The result was later extended by Bhat to semigroups of completely positive maps that act on arbitrary von Neumann algebras [Bha99]. The construction of the dilation has been reformulated in various ways; the one I like is in Chapter 8 of [Arv03] (also see [Arv02]). Another approach, due to Muhly and Solel [MS02], is based on correspondences over von Neumann algebras. The history of earlier approaches to this kind of dilation theory is summarized in the notes of Chapter 8 of [Arv03].

We now state the appropriate result for  $\mathcal{B}(H)$  without proof:

**Theorem 9.2.** *For every normal UCP map  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ , there is a Hilbert space  $K \supseteq H$  and a normal  $*$ -endomorphism  $\alpha : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  satisfying  $\alpha(\mathbf{1}) = \mathbf{1}$  that is a proper dilation of  $\phi$  as in (9.1).*

*Remark 9.3* (Minimality and uniqueness). The notion of minimality for a dilation  $\alpha : \mathcal{B}(K) \rightarrow \mathcal{B}(K)$  of a UCP map  $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$  is described as follows. Again, we identify  $\mathcal{B}(H)$  with the corner  $P\mathcal{B}(K)P$ . We have already pointed out that the projections  $\alpha^n(P)$  increase with  $n$ . However, the sequence of (nonunital) von Neumann subalgebras  $\alpha^n(\mathcal{B}(H))$ ,  $n = 0, 1, 2, \dots$ , neither increases nor decreases with  $n$ , and that behavior requires care. The proper notion of minimality in this context is that the set of all vectors in  $K$  of the form

$$(9.4) \quad \alpha^{n_1}(A_1)\alpha^{n_2}(A_2)\cdots\alpha^{n_k}(A_k)\xi,$$

where  $k = 1, 2, \dots$ ,  $n_k = 0, 1, 2, \dots$ ,  $A_k \in \mathcal{B}(H)$ , and  $\xi \in H$ , should have  $K$  as their closed linear span. Equivalently, the smallest subspace of  $K$  that contains  $H$  and is invariant under the set of operators

$$\mathcal{B}(H) \cup \alpha(\mathcal{B}(H)) \cup \alpha^2(\mathcal{B}(H)) \cup \dots$$

should be all of  $K$ . It is a fact that every minimal dilation is proper, but the converse is false. It is also true that every proper dilation can be reduced in a natural way to a minimal one, and finally, that any two minimal dilations of the semigroup  $\{\phi^n : n \geq 0\}$  are isomorphic in a natural sense.

We also point out that there is a corresponding dilation theory for one-parameter semigroups of UCP maps. These facts are discussed at length in Chapter 8 of [Arv03].

## APPENDIX A. BRIEF ON BANACH \*-ALGEBRAS

Banach  $*$ -algebras (defined at the beginning of Section 3) are useful because they are flexible – it is usually a simple matter to define a Banach  $*$ -algebra with the properties one needs. More importantly, it is far easier to define states and representations of Banach  $*$ -algebras than it is for the more rigid category of  $C^*$ -algebras. For example, we made use of the technique in the proof of Theorem 4.1 and the estimate of Remark 4.3.

On the other hand, it is obviously desirable to have  $C^*$ -algebraic tools available for carrying out analysis. Fortunately one can have it both ways, because every Banach  $*$ -algebra  $\mathcal{A}$  is associated with a unique enveloping  $C^*$ -algebra  $C^*(\mathcal{A})$  which has the “same” representation theory and the “same” state space as  $\mathcal{A}$ . In this Appendix we briefly describe the properties of this useful functor  $\mathcal{A} \rightarrow C^*(\mathcal{A})$  for the category of Banach  $*$ -algebras that have a normalized unit  $\mathbf{1}$ . There are similar results (including Proposition A.1 below) for many nonunital Banach  $*$ -algebras – including the group algebras of locally compact groups – provided that they have appropriate approximate units. A comprehensive treatment can be found in [Dix64].

The fundamental fact on which these results are based is the following (see Proposition 4.7.1 of the text [Arv01] for a proof):

**Proposition A.1.** *Every positive linear functional  $\rho$  on a unital Banach  $*$ -algebra  $\mathcal{A}$  is bounded, and in fact  $\|\rho\| = \rho(\mathbf{1})$ .*

What we actually use here is the following consequence, which is proved by applying Proposition A.1 to functionals of the form  $\rho(a) = \langle \phi(a)\xi, \xi \rangle$ :

**Corollary A.2.** *Every operator-valued positive linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}(H)$  is bounded, and  $\|\phi\| = \|\phi(\mathbf{1})\|$ .*

By a *representation* of a Banach  $*$ -algebra  $\mathcal{A}$  we mean a  $*$ -preserving homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  of  $\mathcal{A}$  into the  $*$ -algebra of operators on a Hilbert space. It is useful to assume the representation is nondegenerate in the sense that  $\pi(\mathbf{1}) = \mathbf{1}$ ; if that is not the case, it can be arranged by passing to the subrepresentation defined on the subspace  $H_0 = \pi(\mathbf{1})H$ . Representations of Banach  $*$ -algebras arise from positive linear functionals (by way of the GNS construction which makes use of Proposition A.1) or from completely positive linear maps (by a variation of Theorem 6.1, by making use of Corollary A.2).

While we have made no hypothesis on the norms  $\|\pi(a)\|$  associated with a representation  $\pi$ , it follows immediately from Proposition A.1 that every representation of  $\mathcal{A}$  has norm 1. Indeed, for every unit vector  $\xi \in H$  and  $a \in \mathcal{A}$ ,  $\rho(a) = \langle \pi(a)\xi, \xi \rangle$  defines a positive linear functional on  $\mathcal{A}$  with  $\rho(\mathbf{1}) = 1$ , so that

$$\|\pi(a)\xi\|^2 = \langle \pi(a)^* \pi(a)\xi, \xi \rangle = \langle \pi(a^*a)\xi, \xi \rangle = \rho(a^*a) \leq \|a^*a\| \leq \|a\|^2,$$

and  $\|\pi(a)\| \leq \|a\|$  follows. It is an instructive exercise to find a direct proof of the inequality  $\|\pi(a)\| \leq \|a\|$  that does not make use of Proposition A.1.

*Remark A.3* (Enveloping  $C^*$ -algebra of a Banach  $*$ -algebra). Consider the seminorm  $\|\cdot\|_1$  defined on  $\mathcal{A}$  by

$$\|a\|_1 = \sup_{\pi} \|\pi(a)\|, \quad a \in \mathcal{A},$$

the supremum taken over a “all” representations of  $\mathcal{A}$ . Since the representations of  $\mathcal{A}$  do not form a set, the quotes simply refer to an obvious way of choosing sufficiently many representatives from unitary equivalence classes of representations so that every representation is unitarily equivalent to a direct sum of the representative ones. It is clear that  $\|a^*a\|_1 = \|a\|_1^2$ . Indeed,  $\|\cdot\|_1$  is a  $C^*$ -seminorm, and the completion of  $\mathcal{A}/\{x \in \mathcal{A} : \|x\|_1 = 0\}$  is a  $C^*$ -algebra  $C^*(\mathcal{A})$ , called the enveloping  $C^*$ -algebra of  $\mathcal{A}$ . The natural completion map

$$(A.1) \quad \iota : \mathcal{A} \rightarrow C^*(\mathcal{A})$$

is a  $*$ -homomorphism having dense range and norm 1. This completion (A.1) has the following universal property: For every representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  there is a unique representation  $\tilde{\pi} : C^*(\mathcal{A}) \rightarrow \mathcal{B}(H)$  such that  $\tilde{\pi} \circ \iota = \pi$ . The map  $\pi \rightarrow \tilde{\pi}$  is in fact a bijection. Indeed, Proposition A.1 is equivalent to the assertion that there is a bijection between the set of positive linear functionals  $\rho$  on  $\mathcal{A}$  and the set of positive linear functionals  $\tilde{\rho}$  on its enveloping  $C^*$ -algebra, defined by a similar formula  $\tilde{\rho} \circ \iota = \rho$ .

One should keep in mind that the completion map (A.1) can have a non-trivial kernel in general, but for many important examples it is injective. For example, it is injective in the case of group algebras – the Banach  $*$ -algebras  $L^1(G)$  associated with locally compact groups  $G$ . When  $G$  is commutative, the enveloping  $C^*$ -algebra of  $L^1(G)$  is the  $C^*$ -algebra  $C_\infty(\hat{G})$  of continuous functions that vanish at  $\infty$  on the character group  $\hat{G}$  of  $G$ .

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